*Electronic Journal of Qualitative Theory of Differential Equations*
2012, No. **62**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

# A New Fixed Point Result and its Application to Existence Theorem for Nonconvex Hammerstein Type Integral Inclusions*

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Abstract. In this paper, a generalization of Nadler's fixed point theorem is presented for $H^+$-type $k$-multi-valued weak contractive mappings. We consider a nonconvex Hammerstein type integral inclusion and prove an existence theorem by using an $H^+$-type multi-valued weak contractive mapping.

*Keywords and Phrases:* Multi-valued contraction map, multi-valued weak contractive map, $H^+$-type multi-valued weak contractive map, Hammerstein type integral inclusion, Fixed point.
*2000 Mathematical Subject Classification :* 47G20, 47H10, 47H20, 54H15, 54H25, 81Q05.

## 1. Introduction

In 1969, Nadler [16] proved a fixed point theorem for the set-valued contractions, which is of fundamental importance in nonlinear analysis. Inspired from the fixed point result of Nadler [16], the fixed point theory of set-valued contraction was further developed in different directions by many authors, in particular, by Reich [20, 21], Mizoguchi and Takahashi [15], Ciric [3], Kaneko [9], Lim [13], Lami Dozo [14], Feng and Liu [5], Klim and Wardowski [10], Suzuki [22], Pathak and Shahzad [17, 18] and many others. For details, see [19]. An interesting application of a consequence of Nadler's fixed point theorem was given in Cernea [2]. For other applications of the same result see, for example, [4] [6], [7], [8], [12] and [19].

## 2. Preliminaries and Definitions

Let $(X, d)$ be a metric space. Let $CB(X)$ and $C(X)$ denote the collection of all nonempty closed and bounded subsets of $X$ and the collection of all compact subsets of $X$, respectively.

For $A, B \in CB(X)$, let

$$H(A, B) = \max\Big\{\rho(A, B), \rho(B, A)\Big\},$$

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$$H^+(A,B) = \frac{1}{2}\Big\{\rho(A,B) + \rho(B,A)\Big\},$$

where $\rho(A,B) = \sup_{x\in A} d(x,B)$ and $d(x,B) = \inf_{y\in B} d(x,y)$. It is well known that $H$ is a metric on $CB(X)$. Such a map $H$ is called *Pompeiu-Hausdorff metric* induced by $d$.

A mapping $T : X \to CB(X)$ is said to be a

- *multi-valued contraction mapping* if there exists a fixed real number $k, 0 < k < 1$ such that
$$H(Tx,Ty) \le k\, d(x,y), \tag{2.1}$$
for all $x, y \in X$.
- *multi-valued weak contractive mapping* if there exists a fixed real number $k, 0 < k < 1$ such that
$$H(Tx,Ty) \le k\ \max\{d(x,y), d(x,Tx), d(y,Ty), [d(x,Ty) + d(y,Tx)]/2\}, \tag{2.2}$$
for all $x, y \in X$.
- *multi-valued quasi-contraction mapping* if there exists a fixed real number $k, 0 < k < 1$ such that
$$H(Tx,Ty) \le k\ \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \tag{2.3}$$
for all $x, y \in X$.

**Proposition 2.1**([18]). $H^+$ is a metric on $CB(X)$.

Notice that the two metrics $H$ and $H^+$ are equivalent [11] since

$$\frac{1}{2}H(A,B) \le H^+(A,B) \le H(A,B).$$

In the light of this equivalence and referring to Kuratowski [11], we conclude that $(CB(X), H^+)$ is complete whenever $(X,d)$ is complete. Indeed, it is a simple consequence of the completeness of the Hausdorff metric $H$. Moreover, $C(X)$ is a closed subspace of $(CB(X), H^+)$.

Notice also that $H^+ : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbf{R}$ is a continuous function. To see this, we observe that the inequality

$$H^+(A,B) \le H^+(A,C) + H^+(C,B)$$

holds for any $A, B, C \in \mathcal{CB}(X)$. Now pick any $(A_0, B_0) \in \mathcal{CB}(X) \times \mathcal{CB}(X)$. Then for a given $\epsilon > 0$, we can choose a positive number $\delta = \frac{\epsilon}{2}$ such that

$$|H^+(A,B) - H^+(A_0,B_0)| \le H^+(A,A_0) + H^+(B_0,B) < \delta + \delta = 2\delta = \epsilon$$

whenever $H^+(A,A_0) < \delta, H^+(B_0,B) < \delta$. This shows that $H^+$ is continuous at $(A_0,B_0)$.

In [16], S. B. Nadler proved the following result, which he announced earlier.

**Theorem 2.2.** Let $(X,d)$ be a complete metric space and $T : X \to CB(X)$ a multi-valued contraction mapping. Then $T$ has a fixed point.

In this paper, we intend to generalize this result by weakening the multi-valued contraction to an $H^+$-type multi-valued weak contractive mapping. Our main result is summarized in Section 3. In Section 4, we consider a nonconvex Hammerstein type integral inclusion and prove an existence theorem by using an $H^+$-type multi-valued weak contractive mapping.

## 3. Main results

We begin our discussion with the following definition.

**Definition 3.1.** Let $(X, d)$ be a metric space. A multi-valued mapping $T : X \to \mathcal{CB}(X)$ is called $H^+$-contraction if
(1) there exists a fixed real number $k$, $0 < k < 1$ such that

$$H^+(Tx, Ty) \leq kd(x, y) \text{ for every } x, y \in X,$$

(2) for every $x$ in $X, y$ in $T(x)$ and $\epsilon > 0$, there exists $z$ in $T(y)$ such that

$$d(y, z) \leq H^+(T(y), T(x)) + \epsilon.$$

In [18], Pathak and Shahzad proved the following result.

**Theorem 3.2.** Every $H^+$-type multi-valued contraction mapping $T : X \to CB(X)$ with Lipschitz constant $0 < k < 1$ has a fixed point.

We now introduce the following definition.

**Definition 3.3.** Let $(X, d)$ be a metric space. A mapping $T : X \to CB(X)$ is called an $H^+$-type multi-valued weak contractive mapping if the condition (2) holds and there exists a fixed real number $k$, $0 < k < 1$ such that

$$H^+(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}, \tag{3.1}$$

for all $x, y$ in $X$.

Now we state and prove our main result.

**Theorem 3.4.** Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ an $H^+$-type multi-valued weak $k$-contractive mapping with $0 < k < 1$. Then $T$ has a fixed point.

*Proof.* Notice first that for each $A, B \in CB(X), a \in A$ and $\alpha > 0$ with $H^+(A, B) < \alpha$, there exists $b \in B$ such that $\max\{d(a, b), d(a, Ta), d(b, Tb), \frac{1}{2}[d(a, Tb) + d(b, Ta)]\} < \alpha$. Now, let $L > 0$ be such that $k < L < 1$. Then

$$H^+(Tx, Ty) < L \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}, \tag{3.2}$$

for any $x, y \in X, x \neq y$.

Now we choose a sequence $\{x_n\}$ recursively in $X$ in the following way. Let $x_0 \in X$ be arbitrary. Fix an element $x_1$ in $Tx_0$. From (2) it follows that we can choose $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H^+(Tx_0, Tx_1) + \epsilon \tag{3.3}$$

In general, if $x_n$ be chosen, then we choose $x_{n+1} \in Tx_n$ such that

$$d(x_n, x_{n+1}) \leq H^+(Tx_{n-1}, Tx_n) + \epsilon. \tag{3.4}$$

Set $\epsilon = (\frac{1}{\sqrt{L}} - 1)H^+(Tx_{n-1}, Tx_n)$. Then from (3.4), it follows that

$$d(x_n, x_{n+1}) \le H^+(Tx_{n-1}, Tx_n) + (\frac{1}{\sqrt{L}} - 1)\, H^+(Tx_{n-1}, Tx_n) = \frac{1}{\sqrt{L}} H^+(Tx_{n-1}, Tx_n).$$

Thus, we have

$$\sqrt{L}\, d(x_n, x_{n+1}) \le H^+(Tx_{n-1}, Tx_n) \tag{3.5}$$

for each $n \in \mathbf{N}$.

Thus, from (3.2) we have

$$\begin{aligned}\sqrt{L}\, d(x_n, x_{n+1}) &< L \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n),\\ &\qquad [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]/2\}\\ &\le (\sqrt{L})^2 \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})/2\}\\ &\le (\sqrt{L})^2 \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]/2\}\\ &= (\sqrt{L})^2 \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}.\end{aligned}$$

It follows that

$$d(x_n, x_{n+1}) < \sqrt{L} \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} \tag{3.6}$$

for each $n \in \mathbf{N}$. Note that if $x_n = x_{n+1}$ for some $n \in \mathbf{N}$, then $x_n = x_{n+1} \in Tx_n$, that is, $x_n$ is a fixed point of $T$ and we are finished. So, we may assume that $d(x_{n+1}, x_n) > 0$ for each $n \in \mathbf{N}$. Suppose that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$ for some $n \in \mathbf{N}$, then inequality (3.6) gives

$$d(x_n, x_{n+1}) < \sqrt{L}\, d(x_n, x_{n+1}),$$

a contradiction. So we must have $d(x_{n-1}, x_n) \ge d(x_n, x_{n+1})$ for each $n \in \mathbf{N}$. Hence, for all $n \in \mathbf{N}$, (3.6) yields

$$d(x_n, x_{n+1}) < c\, d(x_{n-1}, x_n), \tag{3.7}$$

where $c = \sqrt{L}$. Repeating the same argument n-times as in (3.7), we obtain

$$d(x_n, x_{n+1}) < c^n\, d(x_0, x_1). \tag{3.8}$$

It is obvious that $\{x_n\}$ is bounded. Indeed, for any $n \in \mathbf{N}$, we have

$$\begin{aligned}d(x_0, x_n) &\le \sum_{i=0}^{n-1} d(x_i, x_{i+1}) < (1 + c + c^2 + \cdots c^n) d(x_0, x_1)\\ &< (1 + c + c^2 + \cdots) d(x_0, x_1) = \frac{1}{1-c} d(x_0, x_1) < \infty.\end{aligned}$$

Further, by virtue of (3.8), one may observe that $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Assume that $u \notin Tu$, that is, $d(u, Tu) > 0$. Now using (3.2) we have

$$\begin{aligned}\frac{1}{2}\Big\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\Big\} &= H^+(Tx_n, Tu)\\ &< L \max\{d(x_n, u), d(x_n, Tx_n), d(u, Tu), [d(x_n, Tu) + d(u, Tx_n)]/2\}\\ &\le L \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu), [d(x_n, Tu) + d(u, x_{n+1})]/2\},\end{aligned}$$

it follows that
$$\frac{1}{2}\liminf_{n\to\infty}\Big\{\rho(Tx_n,Tu)+\rho(Tu,Tx_n)\Big\}\le L\,d(u,Tu).$$
Since $\lim_{n\to\infty} d(x_{n+1},u)=0$ exists, and
$$d(u,Tu)=\frac{1}{2}[d(u,Tu)+d(Tu,u)]\le\frac{1}{2}[\rho(Tx_n,Tu)+\rho(Tu,Tx_n)]+d(x_{n+1},u),$$
it follows that
$$\begin{aligned} d(u,Tu) &\le \frac{1}{2}\liminf_{n\to\infty}[\rho(Tx_n,Tu)+\rho(Tu,Tx_n)]+\liminf_{n\to\infty} d(x_{n+1},u)\\ &\le L\,d(u,Tu)+\lim_{n\to\infty} d(x_{n+1},u)=L\,d(u,Tu)<d(u,Tu),\end{aligned}$$
a contradiction. This implies that $d(u,Tu)=0$, and, since $Tu$ is closed, it must be the case that $u\in Tu$.

Notice that every multi-valued contraction mapping with respect to Pompeiu-Hausdorff metric $H$ is an $H^+$-type multi-valued weak contractive mapping but the converse implication need not be true. To see this, we have the following example:

**Example 3.5**. Let $X=[-2,2]$ and $d: X\times X\to\mathbf{R}$ be a standard metric. Let $T: X\to CB(X)$ be defined by $Tx=\{\frac{x}{4}\}$, if $x\in[-1,2]$ and $Tx=\{2\}$, otherwise. It is clear that if $x,y\in[-1,2]$ or $x,y\in[-2,-1)$, then
$$H^+(Tx,Ty)\le\frac{1}{4}d(x,y).$$
If $x\in[-1,2]$ and $y\in[-2,-1)$, then we have
$$H^+(Tx,Ty)=\frac{1}{2}[|2-\frac{x}{4}|+|2-\frac{x}{4}|]=|2-\frac{x}{4}|\le 2+\frac{1}{4}=\frac{3}{4}\cdot 3\le\frac{3}{4}\cdot\max\{d(y,Ty),d(x,Tx)\}.$$
It follows that
$$H^+(Tx,Ty)\le k\,\max\{d(x,y),d(x,Tx),d(y,Ty),[d(x,Ty)+d(y,Tx)]/2\}$$
for all $x,y\in X$ and $k\in[\frac{3}{4},1)$. To check the condition (2), we consider the following cases:
Case 1. If $x\in[-2,-1)$, then for any $y\in Tx=\{2\}$, there exists $z\in Ty=\{\frac{1}{2}\}$ such that for any $\epsilon>0$
$$d(y,z)=\frac{3}{2}\le\frac{3}{2}+\epsilon=H^+(Ty,Tx)+\epsilon.$$
Case 2. If $x\in[-1,2]$, then for any $y\in Tx=\{\frac{x}{4}\}$, there exists $z\in Ty=\{\frac{x}{16}\}$ such that for any $\epsilon>0$
$$d(y,z)=\frac{3|x|}{16}\le\frac{3|x|}{16}+\epsilon=H^+(Ty,Tx)+\epsilon.$$
Thus all the conditions of Theorem 3.4 are satisfied. Moreover, $0\in T0=\{0\}$ is a fixed point of $T$.

Notice that the map $T$ does not satisfy the assumptions of Theorem 2.2 and Theorem 3.2. Indeed, for $x = -1$ and $y \to -1$ from the left we have

$$H(T(-1), T(y)) = H^+(T(-1), T(y)) = 2 + \frac{1}{4} > k\, d(-1, y),$$

for all $k \in (0, 1)$.

We also notice that since

$$[d(x, Ty) + d(y, Tx)]/2 \leq \max\{d(x, Ty), d(y, Tx\}$$

for all $x, y \in X$, it follows that every weak contractive mapping is quasi-contraction.

Using the technique of the proof of Theorem 3.4, one can easily prove the following result.

**Theorem 3.6.** Let $(X, d)$ be a complete metric space. Let $T : X \to CB(X)$ be a $H^+$-type $k$-multi-valued quasi-contraction mapping with $0 < k < \frac{1}{2}$. Then, $T$ has a fixed point.

Pathak and Shahzad [18] introduced the class of $H^+$-type nonexpansive mappings

**Definition 3.7.** Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $T : X \to \mathcal{CB}(X)$ is called $H^+$-nonexpansive if

$(1')$ $$H^+(Tx, Ty) \leq \|x - y\| \text{ for every } x, y \in X,$$

$(2')$ for every $x$ in $X, y$ in $T(x)$ and $\epsilon > 0$, there exists $z$ in $T(y)$ such that

$$\|y - z\| \leq H^+(T(y), T(x)) + \epsilon.$$

Applying the main result of this section, we obtain the following result which plays a role in the next section.

**Proposition 3.8.**([18]). Let $(X, d)$ be a complete metric space. Suppose that $T_i : X \to CB(X), i = 1, 2$, are two $H^+$-type multi-valued contraction mappings with Lipschitz constant $L < 1$. Then if $Fix(T_1)$ and $Fix(T_2)$ denote the respective fixed point sets of $T_1$ and $T_2$,

$$H^+(Fix(T_1), Fix(T_2)) \leq \frac{1}{1 - \sqrt{L}} \sup_{x \in X} H^+(T_1 x, T_2 y).$$

# 4. Existence Theorem for Nonconvex Hammerstein Type Integral Inclusions

Let $0 < T < \infty$, $I := [0, T]$ and $\mathcal{L}(I)$ denote the $\sigma$-algebra of all Lebesgue measurable subsets of $I$. Let $E$ be a real separable Banach space with the norm $\|\cdot\|$. Let $\mathcal{P}(E)$ denote the family of all nonempty subsets of $E$ and $\mathcal{B}(E)$ the family of all Borel subsets of $E$.

In what follows, as usual, we denote by $C(I, E)$ the Banach space of all continuous functions $x(\cdot) : I \to E$ endowed with the norm $\|x(\cdot)\|_C = \sup_{t \in I} \|x(t)\|$. Consider the following integral equation

$$x(t) = \lambda(t) + \int_0^T k(t, s)\, g(t, s, u(s))\, ds \ \text{ on } [0, T]. \tag{4.1}$$

Here $\lambda, k$ and $g$ are given functions, where $\lambda(\cdot) : I \to E$ is a function with Banach space value, $k : I \times I \to \mathbf{R}_+ = [0, \infty)$ is a positive real single-valued function, while $g : I \times I \times E \to E$ is a map. Let $p \in [1, \infty)$, $q \in [1, \infty)$, and let $r \in [1, \infty)$ be the conjugate exponent of $q$, that is $1/q + 1/r = 1$. Let $\| \cdot \|_p$ denote the $p$-norm of the space $L^p(I, E)$ and is defined by $\|u\|_p = (\int_0^T \|u(s)\|^p ds)^{1/p}$ for all $u \in L^p(I, E)$. Consider the Nemitsky operator associated to $g, p, q$ and $G : L^p(I, E) \to L^q(I, E)$ given by

$$G(u) = g(t, s, u(s)) \, a.e. \, \text{on } I.$$

Consider the linear integral operator of kernel $k, S : L^q(I, E) \to L^p(I, E)$ given by

$$S(u) = \lambda(t) + \int_0^T k(t, s) u(s) ds \; a.e. \, \text{on } I.$$

Thus the Hammerstein type integral equation (4.1) is transformed into the form

$$x = SG(u), \quad u \in L^p(I, E) \; a.e. \, \text{on } I \tag{4.1$'$}$$

$$u(t) \in F(t, V(x)(t)) \quad a.e. \; (I := [0, T]), \tag{4.2}$$

where $V : C(I, E) \to C(I, E)$ is a given mapping. In the sequel, we also use the following: For any $x \in E$, $\lambda \in C(I, E), \sigma \in L^p(I, E)$, we define the set-valued maps $M_{\lambda,\sigma}(t) := F(t, V(x_{\sigma,\lambda})(t))$, $t \in I, T_\lambda(\sigma) := \{\psi(\cdot) \in L^p(I, E) : \psi(t) \in M_{\lambda,\sigma}(t) \;\; a.e. \; (I)\}$.

In order to study problem (4.1)-(4.2) we introduce the following assumption.

**Hypothesis 4.1.** Let $F(\cdot, \cdot) : I \times E \to \mathcal{P}(E)$ be a set-valued map with nonempty closed values satisfying:

$(H_1)$ The function $k : I \times I \to \mathbf{R}_+$ satisfies that $k(t, \cdot) \in L^r(I)$, and $t \to \|k(t, \cdot)\|_r \in L^p(I)$.
$(H_2)$ The set-valued map $F(\cdot, \cdot)$ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$ measurable.
$(H_3)$ There exists $L(\cdot) \in L^1(I, \mathbf{R}_+)$ such that, for almost all $t \in I, F(t, \cdot)$ is $L(t)$-Lipschitz in the sense that

$$H^+(F(t, x), F(t, y)) \leq L(t) \, \|x - y\| \tag{C1}$$

for all $x, y$ in $E$, and for any $x, y \in X$, $w \in F(t, x)$ and any $\epsilon > 0$, there exists $z \in F(t, y)$ such that

$$\|w - z\|^p \leq H^+(F(t, x), F(t, y)) + \epsilon \tag{C2}$$

and $T_\lambda(\cdot)$ satisfies the condition: For any $\sigma \in L^p(I, E)$, $\sigma_1 \in T_\lambda(\sigma)$ and any given $\epsilon > 0$, there exists $\sigma_2 \in T_\lambda(\sigma_1)$ such that

$$\|\sigma_1 - \sigma_2\|_p \leq H^+(T_\lambda(\sigma), T_\lambda(\sigma_1)) + \epsilon. \tag{C3}$$

$(H_4)$ The mappings $k : I \times I \to \mathbf{R}_+, g : I \times I \times E \to E$ are continuous, $V : C(I, E) \to C(I, E)$ and there exist constants $M_1, M_2, M_3 > 0$ such that

$$\|g(t, s, u_1) - g(t, s, u_2)\| \leq M_1 \|u_1 - u_2\|^p, \;\; \forall u_1, u_2 \in E,$$

$$\|V(x_1)(t) - V(x_2)(t)\| \leq M_2 \|x_1(t) - x_2(t)\|, \;\; \forall t \in I, \forall x_1, x_2 \in C(I, E),$$

and

$$|k(t, s)| \leq M_3 \; \forall \, t, s \in I.$$

It is worth mentioning that the system (4.1)-(4.2) includes a large variety of differential inclusions and control systems.

Assume that $U$ is an open bounded subset of $\mathbf{R}^n$ (or $Y$, a subset of $E$ homeomorphic to $\mathbf{R}^n$) and $U_T = (0,T] \times U$ for some fixed $T > 0$. We say that the partial differential operator $\frac{\partial}{\partial t} + L$ is parabolic if there exists a constant $\theta > 0$ such that $\sum_{i,j=1}^n a^{ij}(t,x)\xi_i\xi_j \geq \theta|\xi|^2$ for all $(t,x) \in U_T, \xi \in \mathbf{R}^n$. The letter $L$ denotes for each time $t$ a second order partial differential operator, having either the divergence form $Lu = -\sum_{i,j=1}^n (a^{ij}(t,x)u_{x_i})_{x_j} + \sum_{i=1}^n b^i(t,x)u_{x_i} + c(t,x)u$ or else the non-divergence form $Lu = -\sum_{i,j=1}^n a^{ij}(t,x)u_{x_i x_j} + \sum_{i=1}^n b^i(t,x)u_{x_i} + c(t,x)u$, for given coefficients $a^{ij}, b^i, c \ (i,j = 1,2,\ldots,n)$.

A family$\{G(t) : t \in \mathbf{R}_+\}$ of bounded linear operators from $X$ into $E$ is a $C_0$-semigroup (also called linear semigroup of class $(C_0)$) on $X$ if
(i) $G(0) =$ the identity operator, and $G(t+s) = G(t)G(s)\ \forall\, t,s \geq 0$;
(ii) $G(\cdot)$ is strongly continuous in $t \in \mathbf{R}_+$;
(iii) $\|G(t)\| \leq Me^{\omega t}$ for some $M > 0$, real $\omega$ and $t \in \mathbf{R}_+$.

**Example 4.2.** Set $k(t,\tau)g(t,\tau,u) = G(t-\tau)u, \Phi(x) = x, \lambda(t) = G(t)x_0$, where $\{G(t)\}_{t\geq 0}$ is a $C_0$-semigroup with an infinitesimal generator $A$. Then a solution of system (4.1)-(4.2) represents a mild solution of

$$x'(t) \in Ax(t) + F(t,x(t)), \qquad x(0) = x_0. \tag{4.3}$$

In particular, this problem includes control systems governed by parabolic partial differential equations as a special case. When $A = 0$, the relation (4.3) reduces to

$$x'(t) \in F(t,x(t)), \qquad x(0) = x_0. \tag{5.4}$$

Denote

$$\Phi(u)(t) = \int_0^T k(t,\tau)g(t,\tau,u(\tau))\,d\tau, \ \ t \in I. \tag{4.5}$$

Then the integral inclusion system (4.1)-(4.2) reduces to the form

$$x(t) = \lambda(t) + \Phi(u)(t) \qquad a.e.\ (I), \tag{S}$$

which may be written in more "compact" form as

$$u(t) \in F(t, V(\lambda + \Phi(u))(t)) \qquad a.e.\ (I).$$

Now we recall the following:

**Definition 4.3.** A pair of functions $(x,u)$ is called a solution pair of integral inclusion system $(S)$, if $x(\cdot) \in C(I,E), u(\cdot) \in L^p(I,E)$ and satisfy relation $(S)$.

For our further discussion, we denote by $S(\lambda)$ the solution set of $(4.1) - (4.2)$.

For given $\alpha \in \mathbf{R}$ we denote by $L^p(I,E)$ the Banach space of all Bochner integrable functions $u(\cdot) : I \to E$ endowed with the norm

$$\|u(\cdot)\|_p = \Big(\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)}\|u(t)\|^p\,dt\Big)^{\frac{1}{p}},$$

where $m(t) = \int_0^t L(s)\, ds,\ t \in I$. For our further discussion, we denote $L = m(T)$.

**Theorem 4.4.** Let Hypothesis 4.1 be satisfied, let $\lambda(\cdot), \mu(\cdot) \in C(I, E)$ and let $v(\cdot) \in L^p(I, E)$ be such that

$$d(v(t), F(t, V(y)(t))) \leq p(t) \qquad a.e. \qquad (I),$$

where $p(\cdot) \in L^p(I, \mathbf{R}_+)$ and $y(t) = \mu(t) + \Phi(v)(t),\ \forall t \in I$.
Then for every $\alpha > 1$, there exists $x(\cdot) \in S(\lambda)$ such that for every $t \in I$

$$\|x(t) - y(t)\| \leq \|\lambda - \mu\|_C + M_1 M_3 e^{\alpha M_1 M_2 M_3 L} \Big[ \frac{1}{\alpha^{\frac{1}{2p}} (\alpha^{\frac{1}{2p}} - 1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C + \frac{\alpha^{\frac{1}{2p}}}{\alpha^{\frac{1}{2p}} - 1} \Big( \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt \Big)^{\frac{1}{p}} \Big]^p.$$

*Proof.* For $\lambda \in C(I, E)$ and $u \in L^p(I, E)$, define

$$x_{u,\lambda}(t) = \lambda(t) + \int_0^T k(t, s)\, g(t, s, u(s))\, ds,\ t \in I.$$

Let us consider that $\lambda \in C(I, E), \sigma \in L^p(I, E)$ and define the set-valued maps

$$M_{\lambda,\sigma}(t) := F(t, V(x_{\sigma,\lambda})(t)),\ t \in I, \tag{4.6}$$

$$T_\lambda(\sigma) := \{\psi(\cdot) \in L^p(I, E) : \psi(t) \in M_{\lambda,\sigma}(t) \ \ a.e.\ (I)\}. \tag{4.7}$$

Further, in view of condition (C3) of Hypothesis 4.1($H_3$), $T_\lambda(\cdot)$ satisfies the condition: For any $\sigma \in L^p(I, E)$, $\sigma_1 \in T_\lambda(\sigma)$ and any given $\epsilon > 0$ there exists $\sigma_2 \in T_\lambda(\sigma_1)$ such that

$$\|\sigma_1 - \sigma_2\|_p \leq H^+(T_\lambda(\sigma), T_\lambda(\sigma_1)) + \epsilon. \tag{4.8}$$

Now we claim that $T_\lambda(\sigma)$ is nonempty, bounded and closed for every $\sigma \in L^p(I, E)$.

It is well known that the set-valued map $M_{\lambda,\sigma}(\cdot)$ is measurable. For example the map $t \to M_{\lambda,\sigma}(t)$ can be approximated by step functions and so we can apply Theorem III. 40 in [1]. As the values of $F$ are closed, with the measurable selection theorem we infer that $M_{\lambda,\sigma}(\cdot)$ is nonempty.

Further, we note that the set $T_\lambda(\sigma)$ is bounded and closed. Indeed, if $\psi_n \in T_\lambda(\cdot)$ and $\|\psi_n - \psi\|_p \to 0$, then there exists a subsequence $\psi_{n_k}$ such that $\psi_{n_k}(t) \to \psi(t)$ for a.e. $t \in I$ and we find that $\psi \in T_\lambda(\sigma)$.

Let $\sigma_1, \sigma_2 \in L^p(I, E)$ be given. Let $\psi_1 \in T_\lambda(\sigma_1)$ and let $\delta > 0$. Consider the following set-valued map:

$$\mathcal{G}(t) := M_{\lambda,\sigma_2}(t) \cap \Big\{ z \in E : \|\psi_1(t) - z\|^p \leq M_1 M_2 M_3 L(t) \int_0^T \|\sigma_1(s) - \sigma_2(s)\|^p\, ds + \delta \Big\}.$$

By (C2), it follows that

$$\begin{aligned}
d^p(\psi_1(t), M_{\lambda,\sigma_2}(t)) &\leq H^+\Big(F(t, V(x_{\sigma_1,\lambda})(t)), F(t, V(x_{\sigma_2,\lambda})(t))\Big) + \epsilon \\
&\leq L(t)\|V(x_{\sigma_1,\lambda})(t)) - V(x_{\sigma_2,\lambda})(t))\| + \epsilon \\
&\leq M_2 L(t)\|x_{\sigma_1,\lambda}(t) - x_{\sigma_2,\lambda}(t)\| + \epsilon \\
&\leq M_2 M_3 L(t)\int_0^T \|g(t,s,\sigma_1(s)) - g(t,s,\sigma_2(s))\|\,ds + \epsilon \\
&\leq M_1 M_2 M_3 L(t)\int_0^T \|\sigma_1(s) - \sigma_2(s)\|^p ds + \epsilon.
\end{aligned}$$

Since $\epsilon$ is arbitrary, letting $\epsilon \to 0$, we deduce that $\mathcal{G}(\cdot)$ is nonempty bounded and has closed values. Further, according to Proposition III.4 in [1], $\mathcal{G}(\cdot)$ is measurable.
Let $\psi_2(\cdot)$ be a measurable selector of $\mathcal{G}(\cdot)$. It follows that $\psi_2 \in T_\lambda(\sigma_2)$ and

$$\begin{aligned}
\|\psi_1 - \psi_2\|_p^p &= \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)}\|\psi_1(t) - \psi_2(t)\|^p dt \\
&\leq \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)}(M_1 M_2 M_3 L(t)\int_0^T \|\sigma_1(s) - \sigma_2(s)\|^p ds)dt \\
&\quad + \delta\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)}dt \\
&\leq \frac{1}{\alpha}\|\sigma_1 - \sigma_2\|_p^p + \delta\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)}dt.
\end{aligned}$$

Since $\delta$ is arbitrary, so letting $\delta \to 0$ we deduce from the above inequality that

$$\|\psi_1 - \psi_2\|_p^p \leq \frac{1}{\alpha}\|\sigma_1 - \sigma_2\|_p^p$$

i.e.,

$$\|\psi_1 - \psi_2\|_p \leq \frac{1}{\alpha^{\frac{1}{p}}}\|\sigma_1 - \sigma_2\|_p.$$

This yields

$$d(\psi_1, T_\lambda(\sigma_2)) \leq \frac{1}{\alpha^{\frac{1}{p}}}\|\sigma_1 - \sigma_2\|_p.$$

Thus, we have

$$\rho(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) = \sup_{\psi_1 \in T_\lambda(\sigma_1)} d(\psi_1, T_\lambda(\sigma_2)) \leq \frac{1}{\alpha^{\frac{1}{p}}}\|\sigma_1 - \sigma_2\|_p. \tag{4.9}$$

Now replacing $\sigma_1(\cdot)$ with $\sigma_2(\cdot)$ and arguing as above, we obtain

$$\rho(T_\lambda(\sigma_2), T_\lambda(\sigma_1)) \leq \frac{1}{\alpha^{\frac{1}{p}}}\|\sigma_1 - \sigma_2\|_p. \tag{4.10}$$

Now adding (4.9) and (4.10) and dividing by 2, we obtain

$$\begin{aligned}
H^+(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) &\leq \frac{1}{\alpha^{\frac{1}{p}}}\|\sigma_1 - \sigma_2\|_p \\
&\leq \frac{1}{\alpha^{\frac{1}{p}}}\max\{\|\sigma_1 - \sigma_2\|_p, d(\sigma_1, T_\lambda(\sigma_1)), d(\sigma_2, T_\lambda(\sigma_2)), \\
&\qquad [d(\sigma_1, T_\lambda(\sigma_2)) + d(\sigma_2, T_\lambda(\sigma_1))]/2\}.
\end{aligned}$$

Hence we conclude that $T_\lambda(\cdot)$ is an $H^+$-type multi-valued weak contractive mapping on $L^p(I,E)$. Next, we consider the following set-valued maps

$$\begin{aligned}\tilde{F}(t,x) &:= F(t,x) + p(t),\\ \tilde{M}_{\lambda,\sigma}(t) &:= \tilde{F}(t, V(x_{\sigma,\lambda})(t)), \qquad t \in I,\\ \tilde{T}_\lambda(\sigma) &:= \{\psi(\cdot) \in L^p(I,E) : \psi(t) \in \tilde{M}_{\lambda,\sigma}(t) \ a.e.\,(I)\}.\end{aligned}$$

It is obvious that $\tilde{F}(\cdot,\cdot)$ satisfies Hypothesis 4.1.
Let $\phi \in T_\lambda(\sigma), \delta > 0$ and define

$$\mathcal{G}_1(t) := \tilde{M}_{\lambda,\sigma}(t) \cap \Big\{z \in X : \|\phi(t) - z\|^p \leq M_2 L(t)\|\lambda - \mu\|_C^p + p(t) + \delta\Big\}.$$

Using the same argument as used for the set valued map $\mathcal{G}(\cdot)$, we deduce that $\mathcal{G}_1(\cdot)$ is measurable with nonempty closed values.

Next, we prove the following estimate:

$$H^+(T_\lambda(\sigma), \tilde{T}_\mu(\sigma)) \leq \frac{1}{\alpha^{\frac{1}{p}} M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}}\|\lambda - \mu\|_C + \Big(\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt\Big)^{\frac{1}{p}}. \tag{4.11}$$

Let $\psi(\cdot) \in \tilde{T}_\mu(\sigma)$. Then

$$\begin{aligned}\|\phi - \psi\|_p^p &= \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} \|\phi(t) - \psi(t)\|^p dt\\ &\leq \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} [M_2 L(t)\|\lambda - \mu\|_C^p + p(t) + \delta] dt\\ &\leq \|\lambda - \mu\|_C^p \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} M_2 L(t) dt\\ &\quad + \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt + \delta \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} dt\\ &\leq \frac{1}{\alpha M_1 M_3}\|\lambda - \mu\|_C^p + \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt\\ &\quad + \delta \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} dt.\end{aligned}$$

Since $\delta$ is arbitrary, so letting $\delta \to 0$ we deduce from the above inequality that

$$\|\phi - \psi\|_p^p \leq \frac{1}{\alpha M_1 M_3}\|\lambda - \mu\|_C^p + \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt.$$

Thus, by taking $\frac{1}{p}$th power on both sides of the above inequality breaking the right hand side, one obtains (4.11).

Now applying Proposition 3.8 we obtain

$$\begin{aligned}H^+(Fix(T_\lambda), Fix(\tilde{T}_\mu)) &\leq \frac{1}{\alpha^{\frac{1}{2p}}(\alpha^{\frac{1}{2p}} - 1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}}\|\lambda - \mu\|_C\\ &\quad + \frac{\alpha^{\frac{1}{2p}}}{\alpha^{\frac{1}{2p}} - 1}\Big(\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt\Big)^{\frac{1}{p}}.\end{aligned}$$

Since $v(\cdot) \in Fix(\tilde{T}_\mu)$, it follows that there exists $u(\cdot) \in Fix(T_\lambda)$ such that

$$\|v-u\|_p \leq \frac{1}{\alpha^{\frac{1}{2p}}(\alpha^{\frac{1}{2p}}-1)M_1^{\frac{1}{p}}M_3^{\frac{1}{p}}}\|\lambda-\mu\|_C + \frac{\alpha^{\frac{1}{2p}}}{\alpha^{\frac{1}{2p}}-1}\Big(\int_0^T e^{-\alpha M_1M_2M_3m(t)}p(t)dt\Big)^{\frac{1}{p}}. \tag{4.12}$$

We define

$$x(t) = \lambda(t) + \int_0^T k(t,s)\,g(t,s,u(s))\,ds.$$

Then one has the following inequality:

$$\begin{aligned}\|x(t)-y(t)\| &\leq \|\lambda(t)-\mu(t)\| + M_1M_3\int_0^T \|u(s)-v(s)\|^p\,ds \\ &\leq \|\lambda-\mu\|_C + M_1M_3e^{\alpha M_1M_2M_3L}\|u-v\|_p^p.\end{aligned}$$

Combining the last inequality with (4.12) we obtain

$$\begin{aligned}\|x(t)-y(t)\| \leq \|\lambda-\mu\|_C + M_1M_3e^{\alpha M_1M_2M_3L}\Big[&\frac{1}{\alpha^{\frac{1}{2p}}(\alpha^{\frac{1}{2p}}-1)M_1^{\frac{1}{p}}M_3^{\frac{1}{p}}}\|\lambda-\mu\|_C \\ &+ \frac{\alpha^{\frac{1}{2p}}}{\alpha^{\frac{1}{2p}}-1}\Big(\int_0^T e^{-\alpha M_1M_2M_3m(t)}p(t)dt\Big)^{\frac{1}{p}}\Big]^p.\end{aligned}$$

This completes the proof.


### Acknowledgments

The authors express their gratitude to the learned referees for their helpful comments on an earlier version of this paper.


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(Received June 17, 2011)